

# Fundamental Slinky Oscillation Frequency using a Center-of-Mass Model

Philip Gash

Physics Department, California State University

Chico, CA 95929-0202

Email: [phil.gash@gmail.com](mailto:phil.gash@gmail.com)

## Abstract

Previously, the Slinky has been modeled as a continuous mass distribution undergoing longitudinal wave motion and the corresponding expression for the fundamental oscillation frequency does not agree with the experimental data as the number of Slinky coils decreases. A center-of-mass Slinky model is introduced which correctly predicts the fundamental oscillation frequency for any number of coils. The model is based on important physics: the restoring force for oscillations depends only upon the coils between the center of mass and the support . The model provides a relationship between its oscillation frequencies and those obtained using the massless spring approximation commonly used in oscillation problems. In terms of the Slinky's physical parameters the model provides a physical definition of the effective mass introduced in the Slinky wave models. The center-of-mass model can be easily understood by first and second year undergraduate physics and engineering students, it involves no calculus since all equations involve only discrete summations .

# 1 Introduction

In previous works dealing with Slinky oscillation frequencies, the Slinky has been modeled as a continuous mass distribution ( hereafter CMD) undergoing longitudinal wave motion which is taken to replicate the Slinky's motion. [1–8] The longitudinal wave treatment requires students to understand how to solve the partial differential wave equation together with boundary and initial conditions. First and second year physics and engineering students have not yet attained that level of proficiency in mathematics. Hence, the question arises is there is a more easily understood Slinky model which allows students to calculate the fundamental Slinky oscillation frequency for a free hanging Slinky ? This paper provides an affirmative answer to that question by using a center-of-mass ( hereafter COM ) model of a Slinky. The model requires no calculus and hence can be easily understood by first and second year engineering and physics student.

The paper is organized as follows: Section 1 is a full introduction to the problem with the massless spring ( hereafter MLS) model which discusses and defines the effective mass. It continues with a discussion of the Slinky's two types of coils. It terminates with a table of the percentage difference between the experimental frequency data and the MLS frequency calculations. Section 2 treats the methods associated with the COM model. They include the rationale for the model, the derivation of equations such as the position of the COM and an expression for the fundamental oscillation frequency. Section 3 for both the MLS and COM models presents the percentage differences with the data for the fundamental frequency and its ratios. Section 4 discusses three features which stress the usefulness of the

mechanical COM model of the Slinky. First, the physical origin of the effective mass coefficient is given a clear physical meaning. Second, a physical justification is given for the MLS model in terms of the COM model's parameters, with examples. And third, an expression is obtained for the Slinky's elastic potential energy, it is computed and compared to the gravitational potential energy. Section 5 concludes with the important features of the COM Slinky model and makes the point that it can be understood and used by first and second year physics and engineering students.

## 1.1 Massless Spring Model

In general, the CMD results are that the Slinky oscillation frequency is dependent upon an effective mass term  $M_{eff}$ . Our interest here is in the Slinky's fundamental oscillation frequency. The expression for the lowest angular frequency  $\omega$  is given by [6, 7]

$$\omega^2 = \frac{K}{M_{eff}}, \quad (1)$$

where  $M_{eff} = \beta M$ . The Slinky mass is denoted by  $M$  and the effective mass coefficient  $\beta = 4/\pi^2$ . Equation (1) represents a Slinky model as an effective mass  $M_{eff}$  connected to a massless spring ( hereafter MLS) where  $K$  is its spring constant due all the spring's coils. Equation (1) is used to calculate the frequency ratios corresponds to different number of free coils ( 60 to 20 in 10 coil decrements) for the MLS model, and they are compared to the experimentally determined ratios. Ratios are used because they eliminate the need for specific spring constant values. The spring constant ratios are dependent only upon the ratio of the number of spring coils.

The investigation is carried out in three steps. First, the fundamental Slinky oscillation frequencies are measured for a different number of free coils. The frequency ratios are computed for all frequencies relative to the frequency for 60 free coils. Second, the corresponding ratios of the MLS model are computed using Eq.(1). Both ratios and the corresponding percentage differences between the experimental and calculated values are presented in Table 1. The percentage differences between the two ratio sets are not random, nor small, and all the experimental ratios are smaller than those of the MLS model. Clearly as the number of free coils is reduced the MLS model's ratios diverge from the experimental value ratios. It is at this point where the longitudinal wave description begins to fail. We seek an easily understood model which accounts for the divergence. Third, a mechanical center-of-mass model ( hereafter, COM model) is introduced to model the Slinky . The COM model accounts for the divergence, the value of the MLS model's effective mass coefficient  $\beta$  and provides some physical insight into the restoring forces producing the oscillations.

## 1.2 The Slinky Details

The Slinky used was a Poof-Slinky, Inc ( [www.poof-slinky.com](http://www.poof-slinky.com) ) of 206 grams with 84 coils. Each coil has a mass of  $2.45 \times 10^{-3}$  kg. and a thickness of 0.7 mm. Two coils are clipped together for support at each end. The two coils on the top do not participate in the oscillatory motion, they are used to support the hanging Slinky. The two coils at the bottom are included in the attached mass. Close inspection of the bottom of the Slinky indicates not all Slinky coils are identical, they fall into

two groups [9]. In one group of 70 free coils, they are visually separated from each other by the weight of the coils below. Near the bottom, a second group of 10 coils are virtually unseparated and we take them to act not as free coils, but as a mass attached to the Slinky.

The reason for the grouping of the several coils below a given coil is that the weight of the several coils is insufficient to expand the given coil a displacement greater than about 1/2 its thickness. In the present case for a given coil  $k_c = 60$  N/m and the coil thickness is 0.7 mm. The force to expand the given coil 1/2 its thickness is about 0.20 N which is the weight of a little more than 8 coils.

The integrity of the grouped coils ( hereafter the attached mass ) is maintained by placing two small zip-ties around them. The set of free coils  $N_{free}$  and bound or attached coils  $N_{att}$  are referred to as the Slinky. In the event the attached mass is not a set of bound coils,  $N_{att}$  is the equivalent number of coils given by  $N_{att} = M_{att}/m_c$ , where  $M_{att}$  is the attached mass and  $m_c$  is the mass of one coil. Only some or all of the free coils may contribute to the restoring force for the oscillations, whereas all the coils, both the free and attached, contribute to the mass of the Slinky.

### 1.3 Fundamental Frequency Data

The Slinky is placed in motion and 10 period data sets were taken of 20 oscillations each. Table 1 presents the number  $N_{free}$  of free (oscillating) coils, the observed angular frequency  $\omega$ , the angular frequency ratios  $\omega/\omega_{60}$  for the experimental data and the corresponding ratios of the MLS model together with the corresponding percentage difference ( %Diff) between the two. The standard deviation for the

angular frequency values did not exceed 0.01 rad/sec.

Equation (1) was used to determine the ratios for the MLS model in the following manner. For coils in series, the reciprocals of the spring constants add to produce the reciprocal of the series-spring constant. [10] So if  $k_c$  is the single coil spring constant then for  $N_{free}$  of them in series, the series-spring constant is  $k_c/N_{free}$ . Using Eq.(1) for the MSL model we obtain

$$\omega_n^2 = \left(\frac{k_c}{N_{free}}\right)\left(\frac{1}{\beta(N_{free} + 10)m_c}\right), \quad (2)$$

where the first factor is the K value in Eq.(1) and the second factor is  $1/M_{eff}$

The corresponding MLS frequency ratio is expressed as :

$$\frac{\omega_n^2}{\omega_{60}^2} = \frac{4200}{N_{free}(N_{free} + 10)}. \quad (3)$$

As previously noted, the ratio is independent of the spring constant value  $k_c$  and the effective mass coefficient  $\beta$ .

## 2 Methods: Rational for the Center-of-Mass Model

The results shown in the %Diff column of Table 1 are sufficient to argue that the mechanical analogue from the CMD wave treatment can not account for the observed frequency ratios as the number of free coils decreases. The objective of this investigation is to find a Slinky model which accounts for the frequency ratios for any number of coils and is easily understood by 1st and 2nd year physics and engineering

Table 1: Free Coils, Frequency, Ratios & MLS Ratios. Column 1 is free Slinky coils, Column 2 is their corresponding experimental frequencies and Column 3 is the square of the frequency ratios with respect to 60 coil frequency. The corresponding ratios for the MLS model are presented in Column 4 . Column 5 is the percentage differences between experimental and the MLS calculated frequency squared ratios.

Free Coils	$\omega(rad/sec)$	$(\omega/\omega_{60})^2$	MLS $(\omega/\omega_{60})^2$	%Diff
20	8.29	5.36	6.96	+30
30	6.15	2.95	3.50	+19
40	4.96	1.92	2.1	+10
50	4.14	1.33	1.40	+5
60	3.58	1	1	-

students.

At this point we turn to a simple but instructive observation from which to construct a simple Slinky mechanical model: For the fundamental frequency mode all the free coils and the attached mass oscillate in phase; hence, the COM oscillates with the same frequency. Therefore, we turn our attention from several oscillating single coil springs and an attached mass to the oscillation of only one mass, that of the COM of the system and we inquire about the restoring force acting upon it to produce oscillations. In equilibrium , the upwards force exerted on the COM is due *only to those coils between the COM and the support point*. The downwards force due to the weight of the remaining coils is constant and does not participate as a restoring force. Hence, it is only these coils between the support and the COM which supply the restoring force for the oscillation.

In fact, this coil assignment is consistent with a Slinky or spring with a heavy attached mass. As the mass attached at or below the bottom of the spring is increased, system's COM migrates further down from the support point; hence, the number of

restoring force coils increases. As soon as the COM is at any point at or below the bottom of the spring, the coil assignment includes all the coils acting to produce the restoring force. This is an implicit assumption in the MLS model and it is a normal assumption made in dealing with the classic case of a oscillating mass attach to a spring.

## 2.1 Center-of-Mass Model's Properties

The COM model is simple in concept and rich in results. The simplicity of the COM model is that it considers a Slinky as a series of one-coil springs. The mathematical treatment here consists of only discrete sums over all or part of the coils. Undergraduate physics students may easily follow the derivations and evaluation of the summations over the coils in the discussions by use of the two summation identities below:

$$\sum_{1 \leq i \leq N} i = (N/2)(N + 1), \quad (4)$$

and

$$\sum_{1 \leq i \leq N} i^2 = (N/6)(N + 1)(2N + 1). \quad (5)$$

The richness of the model is that yields several quantities: (1) the COM's displacement from the top support  $D_{sys}$ , (2) the single coil angular vibration frequency  $\omega_c$  and the single coil spring constant  $k_c$ , (3) a natural length  $\delta$  associated with the Slinky, where all distances can be expressed as some multiple of  $\delta$ , (4) the number of coils (q) between the COM and the support point, (5) a simplified expression for the Slinky's elastic potential energy and (6) the physical origin of the effective mass

coefficient  $\beta$  employed in the MLS model. These quantities are shown in Table 2 for the different number of free-coil Slinkys used in the data acquisition. In addition, the model is used to establish a ratio of two model's  $\omega^2$  values which is dependent upon only the parameters of the COM model.

The input data for the aforementioned quantities is the number of coils  $N$ , the Slinky mass  $M$ , and the Slinky's free hanging length  $L_{free}$ . It is defined as the length from the support to the top of the coils of the attached mass and it is displayed in Fig.(1).

We adopt the scheme initially put forth by Sawicki [11] to model a Slinky where it taken as a series sequence of  $N_{free}$  Hooke's Law coils with each coil supporting the weight of all those below it including any attached mass of  $N_{att}$  coils. Hooke's Law applied to the  $i$ -th free coil yields

$$k_c S_i = m_c g (N_{free} - i) + N_{att} m_c g \quad (6)$$

where  $k_c$  is the spring constant of a coil,  $m_c$  is a coil's mass, and  $S_i$  is the coil's separation between the its top and bottom, as shown in Fig.1.

The displacement  $D_i$  to the bottom of the  $i$ -th coil from the top support is obtained by summing the coil separations  $S_j$  from  $j = 1$  to  $j = i$  to yield after using the first summation identity

$$D_i = \sum_{j=1}^i S_j = (g/\omega_c^2) ((N_{free} + N_{att} - 1/2)i - i^2/2), \quad (7)$$

where  $\omega_c^2 = k_c/m_c$ . The factor  $g/\omega_c^2$  has units of length and therefore it is taken as a

Table 2: COM Model parameters, the free coils  $N_{free}$ , the hanging length  $L_{free}$ , the COM position below the support, the number of coils Q responsible for the restoring force, and the  $Q/N_{free}$  ratio

$N_{free}$	$L_{free}(m)$ ,	COM (m)	Q	$Q/N_{free}$
20	0.158	0.118	12.9	0.65
30	0.279	0.201	15.9	0.53
40	0.476	0.338	21.9	0.55
50	0.616	0.472	22.0	0.44
60	0.876	0.606	27.1	0.45

natural length parameter  $\delta$  associated with the Slinky. The value of  $\delta$  is determined from the Slinky's hanging length  $L_{free}$  by recognizing it as the displacement of the  $N_{free}$ -th coil. We solve Eq.(7) for  $\delta$  which satisfies

$$L_{free} = \delta(N_{free}/2)(N_{free} - 1 + 2N_{att}) \quad (8)$$

with  $N_{free} = 70$  free coils,  $N_{att} = 10$  attached coils and  $L_{free} = 1.23$  m, and obtain  $\delta = 4.0 \times 10^{-4}$  m. By way of observation  $\delta$  appears to be about 1/2 the width of a Slinky coil which has a thickness of  $7 \times 10^{-4}$  m. The angular frequency for one coil  $\omega_c$  is equal to 156 rad/sec or 24.8Hz. The corresponding single coil spring constant  $k_c$  is equal to 60 N/m.

## 2.2 Slinky's Center of Mass

Consider a hanging Slinky where there are  $N_{free}$  free coils, each with a mass  $m_c$  and spring constant  $k_c$ . And attached to the bottom of the free coils is a mass, made up of the  $N_{att}$  unseparated coils. The COM of the coil-mass system  $D_{sys}$  is determined

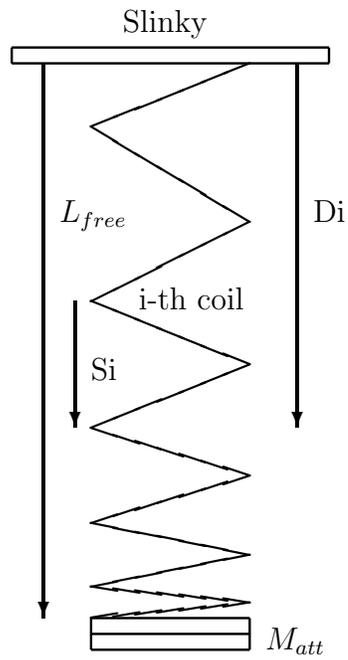


Figure 1: Diagram of a Slinky displaying, the hanging length  $L_{free}$  is measured from the support to the top of the attached mass  $M_{att}$ , the  $i$ -th coil the separation,  $S_i$  is from its top to its bottom and the  $i$ -th coil displacement  $D_i$  is from the support to the bottom of the  $i$ -th coil.

from two quantities: the COM of the expanded free coils  $D_{free}$  and the COM of the attached or bound coils,  $D_{att}$ . The expression for the coil-mass system  $D_{sys}$  is given below:

$$D_{sys} = \frac{m_c N_{free} D_{free} + m_c N_{att} D_{att}}{m_c N_{free} + m_c N_{att}}. \quad (9)$$

The bound or attached coils have a width equal to  $N_{att}$  cw, where cw is the one coil width of 0.7 mm. The location of the COM of the attached coils,  $D_{att}$  is at a displacement from the support point equal to  $L_{free} + N_{att}$  cw /2.

The COM's displacement  $D_{free}$  for the free coil part of the hanging Slinky is obtained by summing the product of each coil's mass  $m_j$  times its displacement  $D_j$  from  $j = 1$  to  $j = N_{free}$  and then dividing by the free coil Slinky's mass  $M = m_c N_{free}$ . After some simplification we obtain

$$D_{free} = \frac{\sum_{j=1}^{N_{free}} m_j D_j}{M} = \delta \frac{(N_{free} + 1)}{2} ((2/3)(N_{free} - 1) + N_{att}). \quad (10)$$

A more convenient parameter is the ratio obtained from the expression for  $D_{free}$  above and  $L_{free}$ , Eq.(8) which yields after some simplification:

$$\frac{D_{free}}{L_{free}} = \left(1 - \frac{1}{N_{free}^2}\right) \frac{\left(\frac{2}{3} + \frac{N_{att}}{N_{free}-1}\right)}{\left(1 + \frac{2N_{att}-1}{N_{free}}\right)}. \quad (11)$$

.As the number of attached coils,  $N_{att}$  gets smaller the remaining 2/3 rds factor is in agreement with previous results [12–14].

In one paper it is argued that  $D_{free} = (3/4)L_{free}$ . [11] This result is inconsistent with the paper's data and the definition of the position of the center of mass. A

straight forward computation using the paper's data yields  $D_{free} = 0.62L_{free}$  which is in good agreement with Eq.( 11) above. Another publication contains a similar misconception [15–17]. Both papers seem to miss the point that the center of mass is that point about which the first moment of the mass is zero. So while the unexpanded Slinky's coil separation is a constant, the coils are symmetric about the center coil which is the COM or the geometrical center. However, the expanded (free hanging) Slinky's coil separation is not constant and hence, the COM location is no longer in the geometrical center.

### 3 Results: Fundamental Oscillation Frequency

For the fundamental oscillation frequency, the important physical point is the COM's restoring force is due only to *the coils between the COM and the support point*. Therefore, the expression for the spring mode angular frequency is expressed by

$$\omega^2 = K_q/M, \tag{12}$$

where  $K_q$  is the spring constant for the series ensemble of  $q$  coils between the COM and the support and  $M$  is the Slinky's mass.

The COM's  $q$  value for the hanging Slinky above is found by solving Eq.(7) where  $q = i$ . As an example, we take the case where  $L_{free} = 1.23$  m,  $N_{free} = 70$  and  $N_{att} = 10$ . First, we need the ratio  $D_{free}/L_{free}$  obtained from Eq.(11), its value is 0.64, hence,  $D_{free} = .78$  m. Second, we need to determine the COM location of the system given by Eq.(9) which yields a value of 0.84 m. We employ Eq.(7) to

Table 3: The experimental frequencies, those calculated from the COM model and the MLS model are displayed as a function of the Slinky free coils. The %Diff for each is calculated with reference to the experimental frequency data.

$N_{free}$	Data $\omega$ (rad/sec)	COM $\omega$ (rad/sec)	% Diff	MLS $\omega$ (rad/sec)	% Diff
20	8.29	7.95	-4	10.0	+21
30	6.15	6.20	+1	7.10	+15
40	4.96	4.73	-5	5.49	+11
50	4.14	4.40	+4	4.48	+8
60	3.58	3.48	-3	3.79	+1

determine the q value, the number of coils between the coil-mass system's COM and the support. The solution is  $q = 36.2$  coils. Do not round up as it may include one coil or a fraction thereof *below* the center of mass. Given that the spring constant for a series of springs adds as reciprocals, we obtain

$$K_q = k_c/q = 60/36.2 = 1.65N/m. \quad (13)$$

Using Eq.(12) the calculated angular frequency for the lowest frequency spring mode is  $\omega = 2.91$  rad/sec corresponding to a period  $T_{cal} = 2.15$  sec. It is in good agreement with the observed value of  $T_{expt} = 2.0$  sec for the Slinky used here.

The procedure above was used to calculate the angular frequency values,  $\omega$  for the COM model and they are presented in Table 3. The additional parameters for the computation are shown in Table 2. The corresponding frequencies for the MLS model were computed using Eq.(2), The percentage difference ( % Diff) of each was computed with reference to the angular frequency data shown in Tables 1. and 3.

The COM model provides physical insight into the effective mass of the MLS

model. Inspection of Table 3 indicates a decreasing difference between the frequencies of the two models as  $N_{free}$  increases. So for large  $N_{free}$  values ( $N_{free} \gg M_{att}$ ) we equate the two expressions for the frequency which yields:

$$\omega^2 = \frac{K}{\beta M} = \frac{K_q}{M}, \quad (14)$$

where  $\beta = 4/\pi^2$  in the MLS model. In the present case we consider  $\beta$  an unknown. In the MLS model  $K = k_c/N_{free}$ , we know now this assumes all free coils produce the restoring force. And for the COM model  $K_q = k_c/q$ . where only  $q$  free coils produce the restoring force. We solve the following equation for  $\beta$ ,

$$\frac{k_c}{N_{free}\beta M} = \frac{k_c}{qM}, \quad (15)$$

which yields

$$\beta = \frac{q}{N_{free}}. \quad (16)$$

The expression for  $\beta$  indicates  $beta < 1$  due to the fact the number of coils supplying the restoring force ( $q$ ) is less than the number of free coils  $N_{free}$  of the Slinky. Second, when the COM is at or below the free coils,  $q = N_{free}$  and thus  $\beta = 1.0$  and we obtain the normal equation for the oscillation frequency:

$$\omega^2 = \frac{K}{M}. \quad (17)$$

We evaluate the  $q/N_{free}$  ratio for large  $N_{free}$  when  $N_{att}$  is negligible. In this case,

Eq.(9) yields

$$D_{sys} = D_{free} = \frac{\delta N_{free}^2}{3}. \quad (18)$$

Now we solve Eq.(7) for  $q$  and obtain  $q = 0.42 N_{free}$ . The ratio becomes 0.42 which compares favorably with the accepted value  $\beta = 0.41$ . ( $\beta = 4/\pi^2$ ) The data from Table 2 shows the  $q/N_{free}$  ratios increase with decreasing  $N_{free}$  up to a value of 0.65 for  $N_{free} = 20$ . In hindsight, since the mass of a spring is fixed, it would be more appropriate in the CDM treatment to associate the  $\beta$  factor with the spring's  $K$  value for the simple reason not all the Slinky coils contribute to the restoring force. The term  $K/\beta = 2.38K$  corresponds to approximately one-half of the coils producing the restoring force.

### 3.1 Slinky's Elastic Potential Energy

The simplicity of the COM model allows computation of the Slinky's elastic potential energy. It is the sum of the elastic potential energy of each coil,  $(1/2)k_c S_i^2$ , taken over only those coils  $Q$  which supports the COM of the coil-mass system. After some simplification of terms, the expression for the Slinky's elastic potential energy is given by

$$\sum_{j=1}^Q (1/2)k_c S_j^2 = (1/2)k_c \delta^2 Q ((N_{free} + N_{att})^2 + (Q + 1)(Q/3 + 1/6 - (N_{free} + N_{att}))). \quad (19)$$

In the case with  $N_{free}=70$  and  $N_{att} = 10$  we have 0.67 J for the elastic potential energy and 1.57 J for the gravitational potential energy with reference to the support point.

### 3.2 Justification of the Massless Spring Approximation

At this point enough formalism has been developed to deal with the ancillary issue of how important in oscillation problems is the inclusion of a spring's mass, given a common first order approximation in dealing with oscillation problems is to neglect the spring's mass.

The starting point is Eq.(12) where the system is divided into two parts:  $M_{free}$  the free coil mass and  $M_{att}$  the bound coils or any other attached mass. The oscillation frequency is given by

$$\omega^2 = \frac{K_Q}{M_{free} + M_{att}}. \quad (20)$$

Here for the Slinky system,  $Q$  represents the number of coils from the support to the COM. The  $Q$  value is obtained by solving Eq.(7) once the system's COM is known.

Equation (21) is rewritten in the following form:

$$\omega^2 = \left(\frac{K_Q}{K_{N_{free}}}\right)\left(\frac{K_{N_{free}}}{M_{att}}\right)\frac{1}{(1 + M_{free}/M_{att})}. \quad (21)$$

The first factor is evaluated using Eq.(13) above to yield  $N_{free}/Q$ . The second factor represents the angular frequency  $\omega_0^2$  used in the MLS approximation. The ratio is given by

$$\frac{\omega^2}{\omega_0^2} = \frac{N_{free}}{Q} \frac{1}{(1 + M_{free}/M_{att})}. \quad (22)$$

Rewriting and recalling  $M_{free} = N_{free}m_c$ , and  $M_{att} = N_{att}m_c$ . The above expression becomes

$$\frac{\omega_0^2}{\omega^2} = \frac{Qm_c}{N_{free}m_c} + \frac{Qm_c}{N_{att}m_c}. \quad (23)$$

The effective mass coefficient  $\beta$  ( see Eq.(14)) can also be rewritten as  $\beta = \omega_0^2/\omega^2$ . In which case, Eq.(23) can serve as a more generalized expression of  $\beta$ .

The ratio of the MLS and COM frequencies has a unique physical interpretation in terms of well defined physics quantities. The ratio is the sum of two fractions: one is the mass of the restoring force coils to the mass of the free coils, and the other is the mass of restoring force coils to the attached mass. A unique feature of the COM model is that it divides the mass of the system into three distinct parts: the mass of all the springs ( $m_c N_{free}$ ), the mass of only the restoring springs ( $Q m_c$ ) and the attached mass ( $N_{att} m_c$ ), with a distinct role played by each.

Equation (22) helps explain why the MLS approximation  $\omega_0^2 = K_{N_{free}}/M_{att}$  is useful in laboratory exercises. Consider a light spring where its mass  $M_{free}$  is 10% of the attached mass  $M_{att}$ , hence the COM is at or very near  $M_{att}$ , thus  $Q \approx N_{free}$ . Using the expression above we find  $\omega_0 = 1.05\omega$ . So for light springs, we find the fractional frequency difference  $(\omega_0 - \omega)/\omega_0$  of 5% is equal to about one-half the ratio of the spring to mass ratio  $M_{free}/M_{att}$  of 10%

Now consider a case where the attached mass is equal to the spring's mass. First, the coil separation  $S_i$  for each coil is augmented by  $M_{a}g/k_c$  due to the additional weight of the attached mass. It is taken into account for the displacement of the i-th coil,  $D_i$  given by Eq.(7). Second, consider a Slinky with 50 free coils and assume the attached mass is a solid circular metal piece which fits inside the 50th coil. The length of the Slinky using Eq.(8) is 1.49 m. From Eq.(11) we obtain  $D_{free} = 0.84$  m and the location of the COM of the system,  $D_{sys} = 1.18$  m determined from Eq.(9). Solving Eq.(7) the COM is location  $Q = 36.5$ . Using Eq. (23) we obtain a fractional

frequency difference of 20 %

We conclude by pointing out there are two different approximations for calculation the oscillation frequency, provided there is not sufficient information to make use of Eq.(20). In both cases, the assumption is the COM of the system is at or below the bottom of the spring. When only the attached mass  $M_a$  and series-spring constant  $K$  are known, then Eq.(12) reduces to the familiar MLS formula,  $\omega^2 = K/M_a$ . When the spring mass  $M_s$  is also known, the second and better frequency approximation from Eq. (19 ) becomes  $\omega^2 = K/(M_a + M_s)$ .

## 4 Conclusion

The %Diff columns in Table 3. present the percentage difference between frequencies and their ratios for both the COM and MLS models with reference to the data values. The MLS model's values for %Diff are clearly increasing as  $N_{free}$  decreases, whereas for the COM model, the %Diff values are smaller and show no systematic trend . We conclude that the COM model is more physically realistic since it accounts for the Slinky frequency for small  $N_{free}$  values. Hence, the COM model fulfills the objective of this investigation. There are two essential physical points which the CMD wave-like approach eludes and are not discussed elsewhere: First, the restoring force exerted by a Slinky is due to only those coils between the COM and the support. Second, the MLS model based upon the CMD introduces an effective mass coefficient,  $\beta = 4/\pi^2$ .The basis of the COM model provides a physical understanding of why there is an effective mass in the MLS model: all the Slinky coils do not contribute

to the restoring force, only a fraction of them do and that fraction ( $q/N_{free}$ ) is equal to the effective mass coefficient  $\beta$ . Finally, the major objective of the paper has been achieved, that is, the simplicity of the center-of-mass Slinky model allows first and second year physics and engineering students to determine the fundamental frequency for a Slinky with any number of coils. There is no need for them to pursue solving a partial differential wave equation or become involved with the notion of a spring's "effective mass"

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