Kinematics of a Falling Slinky using the Center-of-Mass Model

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Abstract

In a recent publication the experimental velocity of the top coil of a falling Slinky exhibited two non-intuitive features of a falling mass. After a short time, there is a maximum speed which then decreases until it is equal that of a point mass falling thru the same distance as the Slinky's center-of-mass. A center-of-mass model is employed to determine the physical reasons for the Slinky's non-intuitive behavior.

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I. INTRODUCTION

A recent publication describes measuring the velocity of a falling Slinky's top coil as a function of time.[1] The two non-intuitive features of the velocity profile are a maximum speed of about 5.6 ± 0.4 m/sec shortly after release. Thereafter the speed decreases at full collapse to that of a point mass falling from the Slinky's center of mass (hereafter COM). The experimental paper's authors used a finite-collapse time model associated with longitudinal wave front propagation to obtain a qualitative curve fit to the velocity profile by adjusting their model's parameters. However, they do not address the physical reasons for the maximum velocity nor the velocity at full collapse.

It is the purpose of this paper to determine the physical process and reasons for the aforementioned two features. The reasons will be discussed in terms of first principles and quantities associated with a coil's motion, such as position, velocity, acceleration, momentum, mass and spring constant. Section II discusses and displays the experimental and calculated velocity profiles. Section III introduces the center-of-mass model and presents the model's parameters of the Slinky.[2] Section IV discusses the kinematics involved in the collapse and their mathematical modeling. Section V concludes with single relationship for the fractional speed difference which explains both the maximum speed and its decrease thereafter. A table presents the time of fall for Slinky's of various lengths which may be useful for student demonestrations.

To preclude numerous repetitive references to the same paper we point out all experimental data may be found in reference 1, by R. C. Cross and M.S.Wheatland, "Modeling a falling slinky", American Journal of Physics, **80**, p. 1051 (2012). in either the text, Tables I, II, and III or numerical data interpolated from Figures 1-11.

II. THE EXPERIMENTAL VELOCITY PROFILE.

Two Slinkys were used in the velocity profile measurement, a standard metal Slinky and a plastic Slinky. We confine this paper to a discussion associated with only the standard metal Slinky. It has a 200 gm mass, a hanging length of 1.0 meters, there are 72 free coils with 8 coils collapsed at the bottom. The Slinky was dropped from rest and photos taken with a camera operating at 300 frames/sec. The coil positions were determined each 0.01 sec, and the velocity's experimental uncertainty is about ± 0.4 m/s .

Two curves are displayed in Fig.1, the open circles are the observed data and the vertical lines represent their experimental uncertainty. Between 0.02 and 0.04 sec, the collapsing coils reach a maximum speed of 5.6 ± 0.4 m/s and there after the speed is diminished until the Slinky is completely collapsed. At collapse, the terminating speed is the $3 \pm .4$ m/s speed which is the range of the 2.7 m/s speed of a point mass falling 0.37 m from the Slinky's COM to its bottom coil. The dark circles are the points from the calculated values described below.

The major feature of Fig.1 is the COM model has the same general behavior as the experimental data: a maximum speed which diminishes until the Slinky is completely collapsed. The difference between the maximum speeds is within the experimental uncertainty of $\pm$ 0.4 m/s. The time of maximum speed from the data is within the interval of 0.02-0.04 sec, and that of the calculated value is within the interval of 0.04-0.05 sec. The experimental time for the collapse was 0.27 sec whereas the COM model's time of collapse was 0.21 sec. The reasons for the difference are discussed in Section V.

III. CENTER-OF-MASS MODEL

The center-of-mass model describes the Slinky as a series of single Hooke's Law coils each with a mass m_c and spring constant k_c as originally proposed by Sawicki[3]. The separation from the top to the bottom of each coil is determined by the weight of the coils below it. All the coils are partitioned into one of two groups,. They are either free with a clear separation between the coils or collapsed near the Slinky's bottom where there is no apparent separation between them. The number of free coils is denoted N_f. N_{att} represents the collapsed coils which are considered a mass attached to the free coils. In the velocity profile computations below $N_f = 72$ and $N_{att} = 8$. The details of the center-of-mass model are fully discussed elsewhere, here we will use only the results. The top-to-bottom separation S_i of the i-th coil is due the weight of the coils below. The i-th coil separation S_i is expressed

$$S_i = \delta((N_f + N_{att} - i), \tag{1}$$

where δ=0.35 mm $=g/\omega_c^2$. The single coil vibration frequency is $\omega_c^2 = k_c/m_c$.

The displacement from the coil support to of the bottom of the j-th coil is D_j and it is given by

$$D_j = \sum_{i=1}^{j} S_i = \delta((N_f + N_{att})j - (j/2)(j+1)). \tag{2}$$

The free hanging length of the Slinky L_f from the top to the bottom of the N_f coils is given by D_{N_f}, and it is used to determine a natural length parameter δ. It is found by solving Eq.(2) with $i = N_f$:

$$L_f = \delta(N_f/2)(N_f - 1 + 2N_{att}), \tag{3}$$

where $\delta = 0.35$ mm when L_f= 1.00 Using the expression $\delta = g/\omega^2$ we find ω= 167 rad/sec. The mass of each coil is 0.0025 kg and thus the single coil spring constant

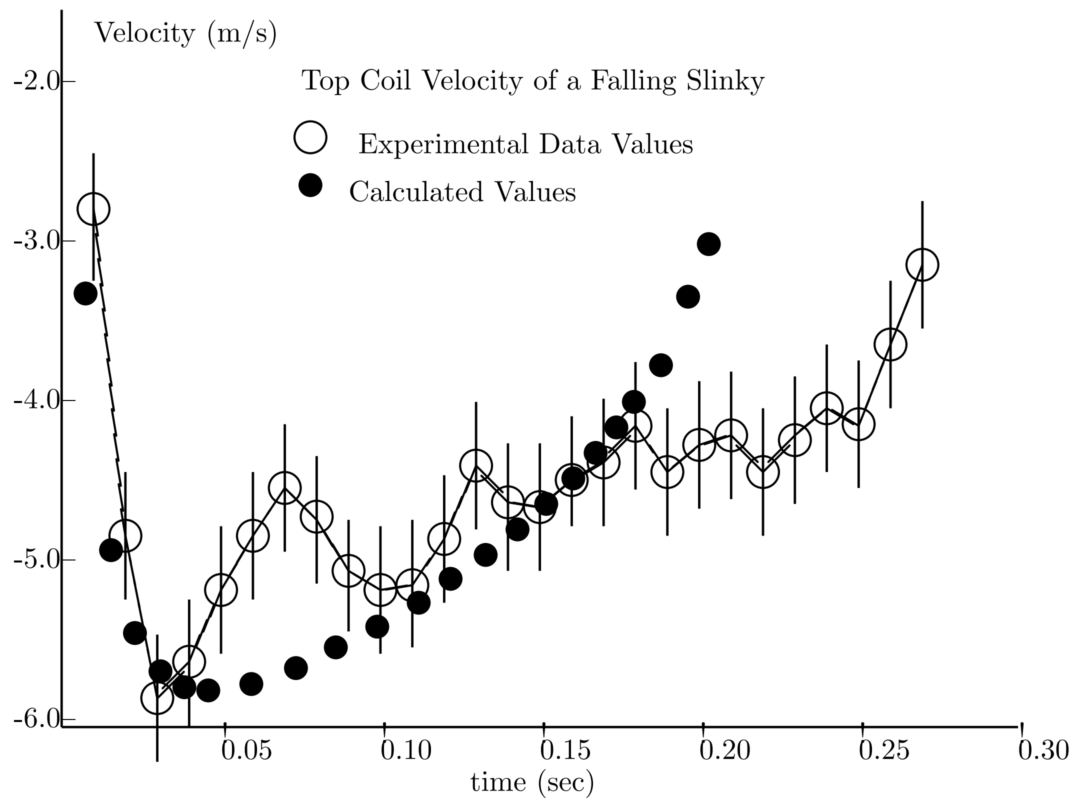

FIG. 1. Velocity profile of the top coil of a falling Slinky. Open circles are data values and black circles are computed values from the COM model. The uncertainty of the experimental velocity is $\pm 0.4 m/s$ and is displayed by the vertical lines.

k_c= 70. N/m. The single coil spring constant determined in the experimental paper for the optimum curve fit is 55 N/m.

IV. CENTER-OF-MASS MODEL'S COLLISION KINEMATICS

During the Slinky's free fall the j collapsed coils on top are colliding with the $N_f - j$ free coils below. In the collision process the j-collapsed coils gain and the lower coils lose one coil. The model for the collapse is as follows: Initially each coil is at rest, when released, the top coil is accelerated thru its separation and collides with the coil below it and this process is repeated until all the coils are collapsed . We define the time during the acceleration and the collision as a collision interval. The collision interval is not fixed as it varies with the coil separation and the subsequent acceleration. In the expressions below the j coil index is also equivalent to a collision interval index. In the computations below, the largest time interval was 4 ms for the fall of the top coil to the one below it. In each collision interval we are only interested in the velocity after the collision, as we take that to be representative of velocity of the top coil since all collapsed coils are moving together. During the collision interval two physical processes occur. Right after the collision when another coil is added to the collapsed coils, the collapsed coils are acted upon by three forces, one due to its weight, one due to the weight of the coils below it, and the third one is the Hooke's Law force due to the expanded free coil below. Applying Newton's 2 Law and taking the downwards direction as positive, we obtain an expression for the acceleration of the j-th coil

$$jm_c a_j = jm_c g + gm_c(N_f + N_{att} - j) + k_c S_j \tag{4}$$

Lets say that j coils are collapsed, its' center-of-mass mass is accelerated a distance X_j to the j+1 coil, a collision then occurs with the j + 1 coil where linear momentum is conserved. Let $v_{0,j}$ be the velocity both after the collision and the initial velocity before the acceleration, then the final velocity $v_{f,j}$ after accelerating thru the center-of-mass displacement X_j is given by

$$v_{f,j}^2 = v_{0,j}^2 + 2a_j X_j, \tag{5}$$

where the acceleration a_j from Eq.(4) is expressed by

$$a_j = (1/j)((jg + g(N_f + N_{att} - j) + \omega_c^2 S_j) = (1/j)(N_f + N_{att})(g + \omega^2\delta) - \omega^2\delta \tag{6}$$

. We assume the acceleration is constant over displacement X_j. In the center-of-mass model, the expression $\omega^2\delta = g$, hence the expression for the acceleration becomes

$$a_j = \frac{2(N_f + N_{att})g}{j} - g. \tag{7}$$

The expression yields the expected acceleration of g after the last coil's collision where $j = N_f + N_{att}$.

In the collision interval the collapsed coil's mass is increased by one coil, and the velocity after the collision is reduced due to momentum conservation. The expression for momentum conservation is written as

$$j m_c v_{f,j} = (j+1) m_c v_{0,j+1} \tag{8}$$

. The velocity of the j+1-th coil after the collision is expressed by

$$v_{0,j+1} = \frac{j}{(j+1)} v_{f,j} \tag{9}$$

.

For example, the first coil is released from rest, so $v_{0,1} = 0$, the center-of-mass undergoes an acceleration of $1558 m/sec^2$ thru a center-of-mass displacement of 0.014 m in 4 ms. Its velocity prior to collision with the second coil is 6.56 m/s, and after the collision the velocity is reduced by 1/2 to 3.28 m/s. The process is then repeated for all the remaining free coils at which time the Slinky is completely collapsed. The COM results displayed by the black circles in Fig.1 are the initial velocity $v_{0,j+1}$ after the collision with the previous coil's final velocity $v_{f,j}$. During the collapsing process the collision interval went from 4 ms for the first coil to about 1 ms for the last coils. The total time for collapse is 0.21 sec, and it is in contrast to the observed time of 0.27 sec.

The fact the bottom of the Slinky remains in suspension while the top collapses upon it is an observation which impresses students and probably gets them to thinking about physics in general. The problem is that with one Slinky the suspension time is too short (2/10 sec.) to easily observe the suspension. The program written to solve the problem above has been executed for two, three four and five 80 coil Slinkys in series Their free hanging length Lf, the maximum speed , the time of suspension, and the number of collapsed coils at maximum speed are tabulated in Table I. The table can be used as a guide in setting up a meaningful demonstration for students about "freely falling objects " in a gravitational field. The suspension times are probably too small given the computed value of 0.21 sec is shorter than the actual measured free fall time of 0.27 sec. for 80 coils.

TABLE I. Suspension Time for 2, 3 , 4 and 5 80-Coil Slinkys in Series

Coils	Lf (m)	$V_{max}(m/s)$	Suspension Time (sec)	V_{max} Coils
160	4.9	12.5	0.41	16
240	10.7	18.7	0.62	22
320	18.6	24	0.82	24
400	29.0	31	1.0	27

V. CONCLUSION

The objective of this investigation is to determine the physical processes for the two non-intuitive features of the collapsing motion. There is one explanation which accounts for both features. The collapsing Slinky is viewed as a falling mass. However the collapsing coil's mass increases as it falls and its speed reaches a maximum and then decreases. Since linear momentum involves both the mass and velocity it is a natural starting point for the analysis,

$$P = MV \tag{10}$$

. We are interested in finding an expression for the incremental speed which is done by taking the total derivative of P, and then dividing it by P to obtain

$$\frac{dP}{P} = \frac{dV}{V} + \frac{dM}{M} \tag{11}$$

The fractional change in the momentum is equal to the sum of the fractional change in the speed plus the fractional change in the mass. All the differentials are replaced by the small increments occurring during one collision interval for the j-th coil . The equation above is rewritten as

$$\frac{\Delta V_j}{V_j} = \frac{\Delta P_j}{P_j} - \frac{\Delta M_j}{M_j} \tag{12}$$

The right side of Eq.(12) was computed for each collapsed j-coil collision with the next free j+1 th coil where ,

$$\frac{\triangle M_j}{M_j} = \frac{M_{j+1} - M_j}{M_j}, \tag{13}$$

and

$$\frac{\triangle P_j}{P_j} = \frac{P_{j+1} - P_j}{P_j}. \tag{14}$$

. In the computation there is sign change in $\triangle V_j$ from positive to negative at the same j value where there is a maximum in the speed V_j.

The sign change and hence the speed maximum can be understood using Eq.(12).. Recall that $P_{j+1} = P_{0,j+1}$, the momentum after the collision with j collapsed coils, and that $P_j = P_{0,j}$. We rewrite their difference by adding in and taking out $P_{f,j}$ to yield

$$P_{j+1} - P_j = (P_{0,j+1} - P_{f,j}) + (P_{f,j} - P_{0,j}) \tag{15}$$

. Since momentum is conserved the first term $(P_{0,j+1} - P_{f,j})$ is zero which leaves

$$P_{j+1} - P_j = P_{f,j} - P_{0,j}. \tag{16}$$

The significance of the equation above is that it is always a positive quantity since $P_{f,j} > P_{0,j}$ due to the acceleration during the collision interval. Clearly ΔM_j is always positive.

Equation (12) is a guide to understanding the shape of the velocity profile. For times less than that of maximum speed the fractional momentum change due to accelerations(large accelerations and large COM displacements) is greater than the fractional mass change hence, the sum is positive and the speed increases. As the collisions progress the fractional momentum decreases (smaller accelerations and smaller COM displacements) and when it equals the fractional mass change, the speed is at its maximum. Afterward the fractional momentum changes are less than the fractional mass changes , so the sum is negative and the speed begins to decrease.

The reason the COM model's 0.21 sec fall time is less than the experimentally observed time of 0.27 sec is attributed to expression used for a coil's acceleration. It is assumed constant and it is not, and it is an over estimate. First, the coil separation S_j decreased to zero as the collapse occurred and second no damping is taken into account. Both terms reduce the computed acceleration and as a consequence increased the length of the collision interval. That in turn would increase the computed falling time.

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[1] R.C.Cross and M.S. Wheatland, "Modeling a falling slinky", Am. J. Phys **80** ,1057 (2012)

[2] Philip Gash, "Slinky Frequency using a Center-Of-Mass Model", see current web site

[3] Mikolaj Sawicki,"Static Elongation of a Suspended Slinky", Phys. Teach. **40**,276-277,(2002)